Sociology 376 Exam 1 Spring 2011 Prof Montgomery

Answer all questions. 260 points possible. You may be time-constrained, so please allocate your time carefully.

[HINT: Somewhere on this exam, it may be useful to know that

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ implies } A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.]$$

1) [50 points] Consider a set of four individuals {1, 2, 3, 4} with the influence matrix

$$W = \begin{bmatrix} .5 & .2 & .3 & 0 \\ 0 & .7 & 0 & .3 \\ 0 & .3 & .4 & .3 \\ 0 & .2 & 0 & .8 \end{bmatrix}$$

where W(i,j) denotes the degree to which i is influenced by j. Further suppose that the initial vector of opinions is given by

$$x_0 = \begin{bmatrix} 7 \\ 5 \\ 8 \\ 4 \end{bmatrix}$$

and that the dynamics of opinion formation are governed by the equation

$$x_t = W\, x_{t-1} \ .$$

a) What opinions will individuals hold at time 1 and time 2?

b) Partition the individuals into communication classes, and then draw the reduced graph of the "influenced by" relation on the set of communication classes. [NOTE: You can use matrix algebra to determine the communication classes and compute the image matrix, but it may be faster to simply determine the reduced graph by inspection.] Which communication classes are open? Which classes are closed?

c) Using your answer to part (b), will all opinions converge in the long run? Briefly explain why or why not. What property of the W matrix led to this result?

2) [80 points] Suppose individuals prefer religion A, religion B, or no religion. (Restated in Markov chain terminology, an individual can be in state A, B, or N). Further suppose that intergenerational mobility between religions is characterized by the transition matrix

$$P = \begin{bmatrix} .8 & 0 & .2 \\ 0 & .7 & .3 \\ .5 & .5 & 0 \end{bmatrix}$$

where the rows and columns are labeled (A, B, N), and P(i,j) indicates the probability that a parent in state i has a child in state j. Initially suppose the reproduction rate (i.e., the average number of children per individual) is equal to one for all individuals regardless of religious preference.

a) Draw the transition diagram for the zero pattern of the P matrix. Is P irreducible? Is P primitive? Explain how you can determine these answers solely by inspection of the zero-pattern transition diagram. What qualitative implications would primitivity of P have for the long-run probability distribution of religious preferences (for an individual's great-great-great-…-grandchildren)?

b) Now suppose the transition matrix is given by

$$P = \begin{bmatrix} .8 & 0 & .2 \\ 0 & 1 & 0 \\ .5 & .5 & 0 \end{bmatrix}$$

Interpret the second row of the P matrix. How would you now classify state B? Given an individual with religious preference A, compute the expected number of generations before his descendants prefer religion B.

c) Given the transition matrix from part (b), now suppose that individuals with religious preference A have (on average) 3 children, while individuals with preference B or N have (on average) only 1 child. Using the row vector x_t to characterize the number of individuals with each religious preference in generation t, give the general equation for x_t as a function of x_0. Further assuming $x_0 = [1\ 1\ 1]$, find x_1 and x_2.

d) In part (c), will all individuals prefer religion B in the long run? Will the population continue to grow? Give some intuition. [HINT: Don't be too quick to apply results you learned in class. While we saw a result of the form "p implies q" where p was the proposition "RP is a primitive matrix," you cannot logically jump to the conclusion that "not p implies not q."]

3) [80 points] Two people choose each period whether to drive on the left-hand side or right-hand side of the road. We can view this process as a Markov chain with 3 states:

(1) both drive on the left
(2) one drives on the left, the other drives on the right
(3) both drive on the right

a) Suppose the transition matrix for the Markov chain is

$$\begin{bmatrix} 1-\varepsilon & \varepsilon & 0 \\ 2/3 & 0 & 1/3 \\ 0 & \varepsilon & 1-\varepsilon \end{bmatrix}$$

Find the limiting distribution over states. In the long run, what proportion of periods are spent in state 1? in state 2? in state 3? [HINT: Your answers will be functions of ε.]

b) Now suppose the transition matrix is

$$\begin{bmatrix} 1-\varepsilon^2 & \varepsilon^2 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & \varepsilon & 1-\varepsilon \end{bmatrix}$$

Again find the limiting distribution. In the long run, what proportion of periods are spent in state 1? in state 2? in state 3? [HINT: Again, your answers will be functions of ε.]

c) Briefly discuss the "macro-level" and "micro-level" interpretations of the limiting distribution in these examples. What terminology does Peyton Young (*Journal of Economic Perspectives* 1998) use to describe the macro-level and micro-level behavior of this model?

d) Briefly define the concept of stochastic stability. What set of states are stochastically stable in part (a)? What set of states are stochastically stable in part (b)? Conceptually, why is it useful to determine the set of stochastically stable states?

e) Suppose we fix ε = .00001. Attempting to find the limiting distribution in part (b) through simulation analysis, a student runs 10,000 chains of length 20, with each chain starting from a randomly chosen state (i.e., 1/3 chance of state 1 or 2 or 3). To estimate the limiting distribution, the student considers the final (period 20) state for each chain, and reports the proportion of chains ending in each state. Approximately what results will the student report? Is this actually the limiting distribution from part (b)? If not, explain why the simulation analysis was faulty, and how you could improve the simulation analysis.

4) [50 points] A population (partitioned into 20-year age classes) has the Leslie matrix below. (Recall the demography convention that L(i,j) reflects population flow *to* age class i *from* age class j.) The eigenvectors and eigenvalues of this matrix are also reported.

```
>> L     % Leslie matrix

L =
         0    0.8000    0.7000         0         0
    0.8000         0         0         0         0
         0    0.9000         0         0         0
         0         0    0.7000         0         0
         0         0         0    0.5000         0

>> [eigvec, eigval] = eig(L)

eigvec =
        0                  0            -0.6724             0.1759 + 0.3019i   0.1759 - 0.3019i
        0                  0            -0.5090             0.0692 - 0.3988i   0.0692 + 0.3988i
        0                  0            -0.4335            -0.4036 + 0.3396i  -0.4036 - 0.3396i
        0             0.0000            -0.2871             0.5347             0.5347
   1.0000            -1.0000            -0.1358            -0.2962 - 0.2493i  -0.2962 + 0.2493i

eigval =
        0                  0                  0                  0                  0
        0                  0                  0                  0                  0
        0                  0             1.0568                  0                  0
        0                  0                  0            -0.5284 + 0.4446i        0
        0                  0                  0                  0            -0.5284 - 0.4446i
```

a) Construct a numerical life table for this population. [HINT: The first column of a life table gives probability of survival (from birth) to each age class; the second column gives expected number of (20-year) periods remaining for each class.]

b) Compute the gross reproduction rate (GRR) and net reproduction rate (NRR) for this population. Is the population growing or shrinking? How do you know?

c) Find the long-run growth rate, and the long-run probability distribution of the population over age classes. Does the population reach this stable growth equilibrium for *any* initial condition? Briefly explain.

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1a) [10 pts] $x_1 = W\,x_0 = \begin{bmatrix} 6.9 \\ 4.7 \\ 5.9 \\ 4.2 \end{bmatrix}$ and $x_2 = W\,x_1 = \begin{bmatrix} 6.16 \\ 4.55 \\ 5.03 \\ 4.3 \end{bmatrix}$

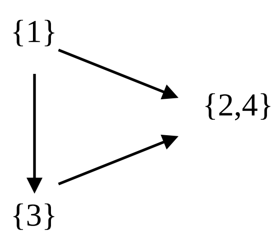
b) [25 pts] Communication classes are the equivalence classes generated by the compound relation "can reach and be reached by." Using the zero-pattern matrix

$$Z = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$\text{reachability}(Z) = (I + Z + Z^2 + Z^3) > 0 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$\text{reachability}(Z)\ \&\ \text{reachability}(Z)' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

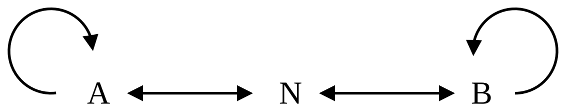
The communication classes (determined by the unique rows of this matrix) are {1}, {2,4}, and {3}. The reduced graph is given by



Classes {1} and {3} are open, while class {2,4} is closed.

c) [15 pts] Given that class {2,4} is closed, it is not influenced by outsiders. Thus, we can solve for the equilibrium opinion vector for that class in isolation. Class {3} is influenced only by {2,4} so will adopt the opinion of {2,4}. Class {1} is influenced by {2,4} and {3} and so will also adopt this opinion. Thus, all opinions will converge. This result occurs because the influence matrix is "centered" – there is only one closed communication class and it has a primitive submatrix.

2a) [25 pts]



P is irreducible and primitive. Using the graph, P is irreducible because every node can reach every other node (directly or indirectly). P is primitive because there is a loop. Primitivity of P implies a unique limiting distribution; every element of the limiting distribution is positive; the limiting distribution does not depend on initial conditions (the individual's initial religious preference).

b) [25 pts] The second row implies that, once an individual prefers religion B, his/her children (and grandchildren and great-grandchildren…) will prefer religion B. That is, B is an absorbing state. Since the state A and N remain non-absorbing, the Q matrix (characterizing transitions from non-absorbing states to non-absorbing states) is given by

$$Q = \begin{bmatrix} .8 & .2 \\ .5 & 0 \end{bmatrix}$$

$$N = I + Q + Q^2 + \ldots = (I\text{-}Q)^{-1} = \left(\begin{bmatrix} .2 & -.2 \\ -.5 & 1 \end{bmatrix}\right)^{-1} = \frac{1}{.2-.1}\begin{bmatrix} 1 & .2 \\ .5 & .2 \end{bmatrix} = \begin{bmatrix} 10 & 2 \\ 5 & 2 \end{bmatrix}$$

The sum of the first row of N (= 10 + 2 = 12) is the expected number of generations before absorption given initial state A.

c) [15 pts] The dynamics are given by the equation

$$x_t = x_0(RP)^t \quad \text{where } R = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and thus } RP = \begin{bmatrix} 2.4 & 0 & .6 \\ 0 & 1 & 0 \\ .5 & .5 & 0 \end{bmatrix}$$

x_1 = [1 1 1] * RP = [2.9 1.5 .6]
x_2 = [2.9 1.5 .6] * RP = [7.26 1.8 1.74]

d) [15 pts] Although B is an absorbing state, individuals in state A have more than enough children to replace themselves even allowing for intergenerational mobility. Note that RP(A,A) = 2.4. This means that each individual in A averages 2.4 children in A. The number of individuals in state A – and the population overall – will grow indefinitely. [If you had access to Matlab, you can show that the population will converge to a stable growth equilibrium with growth rate $\lambda = 2.5191$ and limiting distribution [.7595 .0595 .1809].]

3a) [15 pts] To find the limiting distribution, we can solve the following system:

$$x(1) = (1-\varepsilon)\, x(1) + (2/3)\, x(2)$$
$$x(2) = \varepsilon\, x(1) + \varepsilon\, [1 - x(1) - x(2)]$$

which (after some algebra) yields

$$x(1) = 2 / [3(1+\varepsilon)]$$
$$x(2) = 3\varepsilon / [3(1+\varepsilon)] = \varepsilon / (1+\varepsilon)$$
$$x(3) = 1 - x(1) - x(2) = 1 / [3(1+\varepsilon)]$$

3b) [15 pts] The system of equations is now

$$x(1) = (1 - \varepsilon^2)\, x(1) + (1/2)\, x(2)$$
$$x(2) = \varepsilon^2\, x(1) + \varepsilon\, [1 - x(1) - x(2)]$$

which (after some algebra) yields

$$x(1) = 1 / [1 + \varepsilon + 2\varepsilon^2]$$
$$x(2) = 2\varepsilon^2 / [1 + \varepsilon + 2\varepsilon^2]$$
$$x(3) = \varepsilon / [1 + \varepsilon + 2\varepsilon^2]$$

3c) [10 pts] At the micro level (one pair of individuals), the system would remain in one convention (LL or RR) for a long time, but occasionally "flip" to the other convention. In Young's terminology, we would observe "local conformity" with "punctuated equilibria." At the macro level (across many pairs, assuming each pair is independent of every other), we would observe a stable distribution across conventions, with some proportion of pairs in the LL convention and the remainder in the RR convention. In Young's terminology, there is "global diversity."

3d) [20 pts] State i is stochastically stable when, as ε becomes small, this state has positive probability in the limiting distribution (i.e., $x(i) > 0$). As $\varepsilon \to 0$, the limiting distribution in part (a) becomes [2/3 0 1/3]. Thus, both states 1 and 3 are stochastically stable. As $\varepsilon \to 0$, the limiting distribution in part (b) becomes [1 0 0]. Thus, only state 1 is stochastically stable. Conceptually, when "mistakes" are extremely rare, the Markov chain almost never occupies states that are not stochastically stable. Thus, stochastic stability allows us to determine whether a particular convention is empirically plausible.

3e) [20 pts] Given that ε is very small and that the length of each chain is very short, each chain which begins in state 1 or 3 is likely to remain in the initial state for the entire 20 periods. Chains which begin in state 2 will transition immediately to either state 1 or 3 (with 50% chance) and are then likely to remain in this state. Thus, the outcome of the simulation exercise will be determined entirely by the distribution over initial conditions. The student will report that approximately half of the chains ended in state 1 and that the other half ended in state 3. Note that [1/2 0 1/2] is not the correct limiting distribution from part (b). Given ε very small, the student should have run much longer chains to get the correct results. The problem is not the number of chains (10,000 would seem adequate) but rather the short chain length.

4a) [20 pts] The fundamental matrix is given by

$$N = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ s_1 & 1 & 0 & 0 & 0 \\ s_1 s_2 & s_2 & 1 & 0 & 0 \\ s_1 s_2 s_3 & s_2 s_3 & s_3 & 1 & 0 \\ s_1 s_2 s_3 s_4 & s_2 s_3 s_4 & s_3 s_4 & s_4 & 1 \end{bmatrix}$$

which can be computed with a calculator. Or, if you had access to Matlab,

```
>> S = L; S(1,:) = [0 0 0 0 0];  N = eye(5) + S + S^2 + S^3 + S^4

N =
   1.0000        0         0         0         0
   0.8000    1.0000        0         0         0
   0.7200    0.9000    1.0000        0         0
   0.5040    0.6300    0.7000    1.0000        0
   0.2520    0.3150    0.3500    0.5000    1.0000
```

The first column of the life table is the first column of N. The elements of the second column of the life table are given by the column sums of N. Thus, the life table is

```
>> lifetable = [(1:5)' N(:,1) sum(N)']

lifetable =
   1.0000    1.0000    3.2760
   2.0000    0.8000    2.8450
   3.0000    0.7200    2.0500
   4.0000    0.5040    1.5000
   5.0000    0.2520    1.0000
```

b) [15 pts] GRR = .8 + .7 = 1.5. NRR = (.8)(.8) + (.8)(.9)(.7) = 1.144. Given NRR > 1, the population is growing.

c) [15 pts] The largest eigenvalue of L is growth rate, and the associated eigenvector (normalized to be a probability vector) is the long-run probability distribution. Using the Matlab computations provided, the growth factor is 1.0568 (i.e., the growth rate is 5.68%). The limiting distribution is found by dividing each of the elements of the leading eigenvector by the sum of the elements (= -2.0378). Thus, the distribution given by (the transpose of) the vector [0.3300 0.2498 0.2127 0.1409 0.0667]. For this example, individuals in the 2 oldest age classes don't have children. If the initial distribution was composed entirely of such individuals, the population would die out. Otherwise, the population will reach the stable growth equilibrium described above.

Sociology 376 **Exam 2** **Spring 2011** **Prof Montgomery**

Answer all questions. 240 points possible.

[HINT: Somewhere on this exam, it may be useful to know that the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

has eigenvalues $\lambda_1 = (1/2)(a + d + \text{sqrt}(a^2 + 4bc - 2ad + d^2))$
$\lambda_2 = (1/2)(a + d - \text{sqrt}(a^2 + 4bc - 2ad + d^2))$]

1) [80 points] This semester, we learned to use phase diagrams in the context of nonlinear systems. But phase diagrams can also be used when the system is linear. In particular, consider one of the simple intergenerational social mobility examples we studied early in the course. Dynamics were given by $x_{t+1} = x_t M$ where x is a 1×3 row vector, x(i) is the proportion of the population in social class i, and M is the 3×3 probability transition matrix

$$M = \begin{bmatrix} .6 & .4 & 0 \\ .3 & .4 & .3 \\ 0 & .7 & .3 \end{bmatrix}$$

To simplify notation, we may adopt the notation x = [x(1) x(2) x(3)] = [p 1-p-q q] so that p is the share of the population in class 1, 1-p-q is the share of the population in class 2, and q is the share of the population in class 3.

a) Rewrite the social mobility model as a 2-equation system, with p_{t+1} as a function of p_t and q_t, and q_{t+1} as a function of p_t and q_t.

b) Rewrite the pair of equations from part (a) in “delta” notation, with Δp as a function of p and q, and Δq as a function of p and q.

c) Solve for the p-nullcline and q-nullcline. Then solve (algebraically) for the equilibrium (p*, q*) determined by the intersection of the nullclines.

d) Should the phase diagram be plotted on the unit square (with $0 \le p \le 1$ and $0 \le q \le 1$) or a triangular simplex (with $0 \le p+q \le 1$)? Briefly explain why. Then plot the nullclines on the appropriate diagram. [HINT: Your graph doesn’t need to be perfect, but should be qualitatively correct and properly labeled.]

e) Use the pair of equations from part (b) to determine the dynamics in each region of the phase diagram. Then add arrows to your phase diagram to indicate these dynamics. Is the equilibrium stable? How can you tell from the diagram?

2) [80 points] Consider a public school district where parents have either high income or low income. Each parent has one child, and must decide whether to send the child to the public school or to a private school. Low-income parents can't afford private school tuition, and so always send their children to the public school. High-income parents will send their children to private school if the ratio of low-income to high-income children becomes too high. To be more precise, let H and L denote the numbers of high-income and low-income children attending the public school. Each high-income parent i has tolerance level θ_i, and will send her child to the public school if and only if $\theta_i > L/H$. Tolerance levels of high-income parents are uniformly distributed between 0 and 4. Finally, suppose that the total number of high-income children is equal to 100.

a) Suppose high-income parents have adaptive expectations. Write the equation describing the dynamics of public-school attendance. [HINT: Your equation should give H_{t+1} as a function of H_t and L. Recall that L is fixed. Make sure your equation holds for every value of H_t between 0 and 100.]

b) Suppose L = 50. Plot the cobweb diagram. Solve for the equilibrium (or equilibria) and indicate whether each is stable. What range of initial conditions are associated with each stable equilibrium (i.e., what are the basins of attraction)? [HINT: Your cobweb diagram doesn't need to be perfect, but I am looking for numerical solutions for the equilibria.]

c) Suppose that school district boundaries are redrawn. There are still 100 high-income children but now 80 low-income children. Redo the analysis from part (b), plotting the new cobweb diagram, solving for the equilibrium, and indicating stability.

d) If the school board wants at least some high-income children to attend the public school, what is the maximum number of low-income children that can be placed in the district? Using terminology from dynamical systems, describe how the equilibrium (or equilibria) change as L increases from 50 to 80 and beyond.

3) [80 points] Consider the following two-player game between police (who can choose a high or low level of enforcement) and protestors (who can choose a high or low level of activity). Following the usual convention in game theory, the pairs of payoffs are stated as (row player's payoff, column player's payoff).

protestors

police		H	L
	H	0, -1	-1, 0
	L	-2, 2	0, 0

To analyze this game using the replicator dynamics, we can assume **two** populations of players (the row population from which police are drawn; the column population from which protestors are drawn). The equations for the replicator dynamics may written as

$$\Delta x = \text{diag}(x)\,[Ay - x'Ay]$$
$$\Delta y = \text{diag}(y)\,[Bx - y'\,Bx]$$

where

$$A = \begin{bmatrix} 0 & -1 \\ -2 & 0 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix} \quad x = \begin{bmatrix} p \\ 1-p \end{bmatrix} \quad y = \begin{bmatrix} q \\ 1-q \end{bmatrix}$$

Substitution of A, B, x, and y into the Δx and Δy equations yields the pair of equations

$$\Delta p = p\,(1-p)\,(3q-1)$$
$$\Delta q = q\,(1-q)\,(2-3p)$$

[NOTE: For simplicity, I have set period length is equal to 1 (otherwise, there would be an additional "h" term on the right-hand side of each equation).]

a) Interpret the Δx and Δy equations, explaining the functional form. What is the interpretation of p and q?

b) Plot the p-nullcline(s) and q-nullcline(s) on a phase diagram. Report all steady states (p^*, q^*). How do these compare to the Nash equilibria of the two-player game? Briefly explain why the replicator dynamics may have steady states that are not Nash equilibria.

c) Finish constructing the phase diagram by drawing arrows to indicate the direction of dynamics in each region. [NOTE: Make sure your phase diagram is well labeled.]

d) Is the interior steady state stable? [NOTE: To receive full credit, you must support your answer in one of the two following ways: (1) choose an initial condition close to the interior equilibrium, and compute a trajectory for at least 5 subsequent periods, or (2) derive the Jacobian matrix and compute its eigenvalues. Either way, you might first want to rewrite the Δp and Δq equations in the form $p_{t+1} = g_1(p_t, q_t)$ and $q_{t+1} = g_2(p_t, q_t)$.]

Sociology 376 Exam 2 Spring 2011 Solutions

1a) [20 pts]

$$[p_{t+1} \;\; 1\!-\!p_{t+1}\!-\!q_{t+1} \;\; q_{t+1}] = [p_t \;\; 1\!-\!p_t\!-\!q_t \;\; q_t] \begin{bmatrix} .6 & .4 & 0 \\ .3 & .4 & .3 \\ 0 & .7 & .3 \end{bmatrix}$$

implies $p_{t+1} = p_t\,(.6) + (1 - p_t - q_t)(.3) = .3 + .3\,p_t - .3\,q_t$

$q_{t+1} = (1 - p_t - q_t)\,(.3) + q_t\,(.3) = .3 - .3\,p_t$

b) [10 pts]

$p_{t+1} - p_t = .3 - .7\,p_t - .3\,q_t$ implies $\Delta p = .3 - .7p - .3q$

$q_{t+1} - q_t = .3 - .3\,p_t - q$ implies $\Delta q = .3 - .3p - q$

c) [20 pts]

p-nullcline: $\Delta p = 0$ implies $q = 1 - (7/3)\,p$

q-nullcline: $\Delta q = 0$ implies $q = .3 - .3\,p$

The intersection of the nullclines is determined by

$1 - (7/3)\,p = .3 - .3\,p$ which implies $p^* = 21/61 = .3443$
$q^* = 12/61 = .1967$

d) [15 pts] The phase diagram should be plotted on the triangular simplex. Because p and q are population shares, the sum p+q cannot exceed 1. The equation p+q = 1 defines the hypotenuse of the simplex. Along that edge, everyone in the population belongs to social class 2.

See below for the phase diagram.

e) [15 pts]

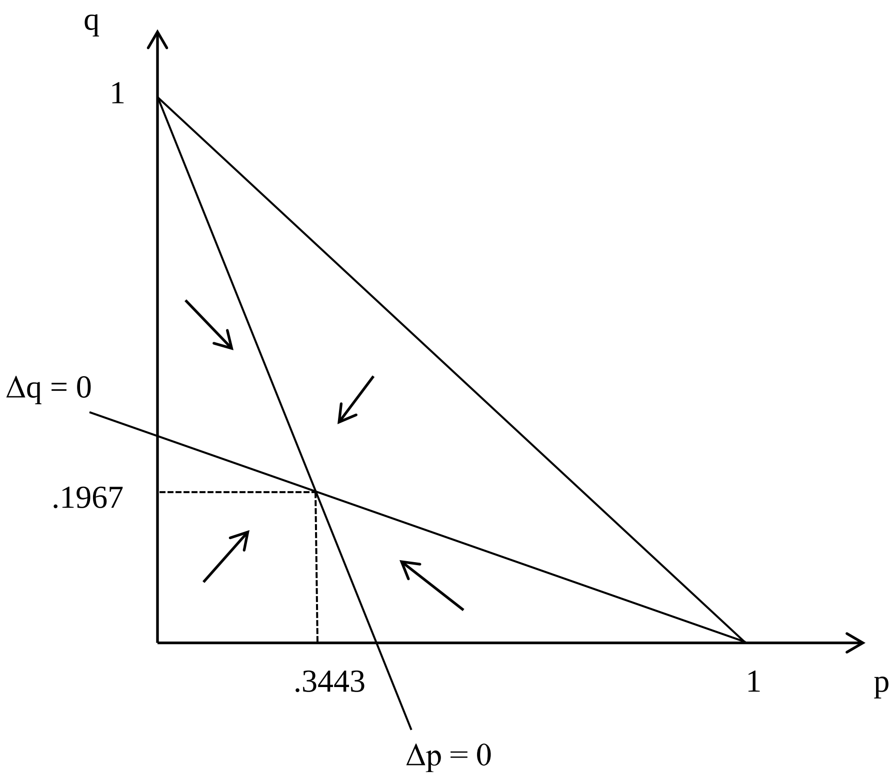
$\Delta p > 0$ implies $q < 1 - (7/3)p$

$\Delta q > 0$ implies $q < .3 - .3p$

See below for phase diagram.

The equilibrium (p*, q*) appears to be stable because all arrows are pointing into the equilibrium.

1 continued) phase diagram for parts (d) and (e)



2a) [20 pts] Assuming L/H is between 0 and 4, the proportion of high-income parents sending their children to public school is (1/4)(4 – L/H). [Parents with tolerances between L/H and 4 send their children to public school; (1/4) is the height of the density function.] Thus, the number of high-income parents sending their children to public school is (100)(1/4)(4 – L/H). The dynamics are thus given by

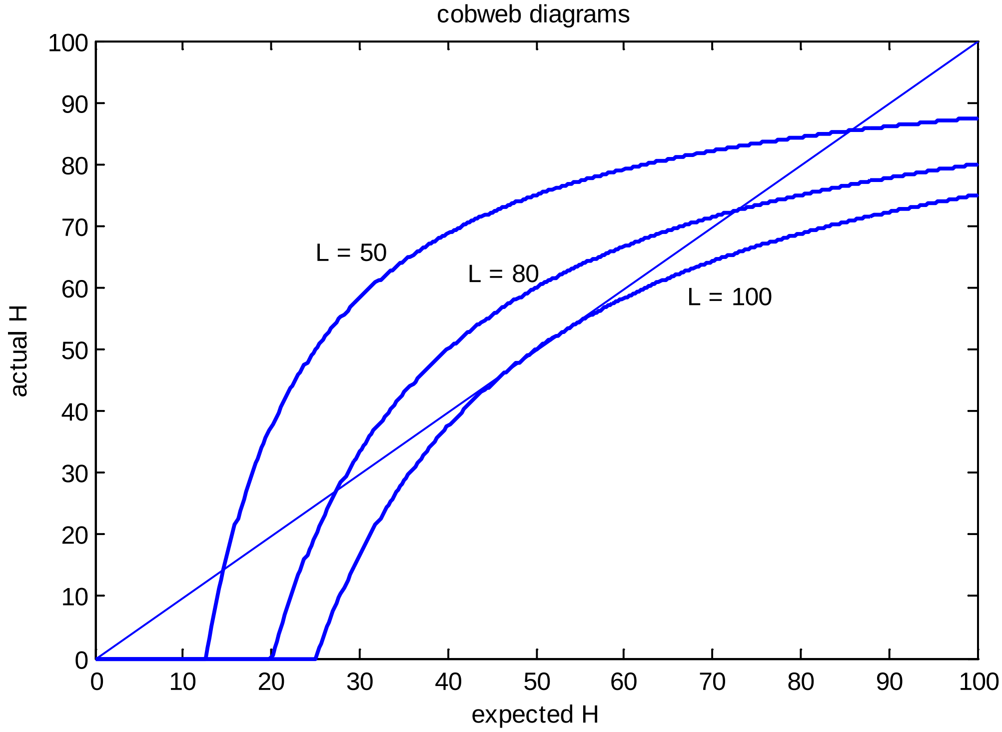
$$H_{t+1} = 100 - 25\, L / H_t$$

But note that no parents send their children to public school when L/H > 4. Thus, restating the dynamics more precisely,

$$\begin{aligned} H_{t+1} &= 100 - 25\, L / H_t && \text{for } H_t \geq L/4 \\ &= 0 && \text{for } H_t < L/4 \end{aligned}$$

Equivalently, $H_{t+1} = \max\{0, 100 - 25\, L / H_t\}$

b) [30 pts]



The interior equilibria are determined by the equation

$$H = 100 - 25\ L/H$$
$$H^2 - 100\ H + 25\ L = 0$$
$$H = (100 \pm \text{sqrt}[(100)^2 - 4\ (1)\ (25\ L)])\ /\ 2$$

For L = 50, this yields H = 85.3 or H = 14.6

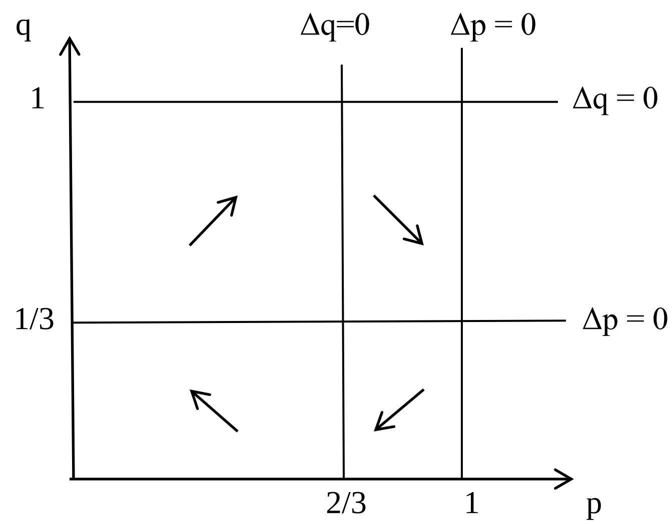
From the diagram, it is apparent that the equilibria at H* = 0 and H* = 85.3 are stable; the equilibrium at H* = 14.6 is unstable. H_t will converge to H* = 85.3 for any initial condition $H_0 > 14.6$; H_t will converge to H* = 0 for any initial condition $H_0 < 14.6$.

c) [20 pts] See the cobweb diagram above. Given L = 80, the interior equilibria are given by H* = 72.4 and H* = 27.6. H_t will converge to H* = 72.4 for any initial condition $H_0 > 27.6$; H_t will converge to H* = 0 for any initial condition $H_0 < 27.6$.

d) [10 pts] If the school board wants to retain any high-income children, it cannot allow L > 100. As shown on the cobweb diagram, an increase in L causes a downward shift in the generator function. If L increases beyond 100, there is a *catastrophe*. Instead of three equilibria, there is a unique equilibrium at H* = 0.

3a) [15 pts] The replicator dynamics assume that actions become more popular when they yield above-average payoffs, and become less popular when they yield below-average payoffs. The Δx equation reflects the change in the distribution of actions in the police (row) population. From the perspective of the police, the expected payoff to each action is given by Ay (where y is the distribution of protestors across actions), and the average payoff for all police is given by $x'Ay$ (where x is the distribution of police across actions). Thus, action i is becoming more popular among police when $(Ay)(i) - x'Ay$ is positive. The Δy equation, which reflects change in the protestor (column) population, can be interpreted similarly. Note the p is the proportion of police choosing high enforcement, while q is the proportion of protestors choosing high activity.

b) [23 pts] $\Delta p = 0$ implies p = 0 or p = 1 or q = 1/3. $\Delta q = 0$ implies q = 0 or q = 1 or p = 2/3.



Recognizing that the horizontal axis is a q-nullcline, and that the vertical axis is a p-nullcline, there are 5 steady states: ($p^* = 0$, $q^* = 0$), ($p^* = 0$, $q^* = 1$), ($p^* = 1$, $q^* = 0$), ($p^* = 1$, $q^* = 1$), and ($p^* = 2/3$, $q^* = 1/3$). The latter is a mixed-strategy Nash equilibrium of the two-player game. The other steady states are not Nash equilibria. Given the functional form for the replicator dynamics, any action which is initially "zeroed out" will never enter the population.

c) [12 pts] From the Δp and Δq equations, we see that $\Delta p > 0$ when $q > 1/3$, and that $\Delta q > 0$ when $p < 2/3$. This implies the arrows shown in the phase diagram above.

d) [30 pts] The interior steady state is unstable. Given $g_1(p,q) = p + p(1-p)(3q-1)$ and $g_2(p,q) = q + q(1-q)(2-3p)$, the Jacobian matrix is given by

$$J = \begin{bmatrix} \partial g_1/\partial p & \partial g_1/\partial q \\ \partial g_2/\partial p & \partial g_2/\partial q \end{bmatrix} = \begin{bmatrix} 1+(1-2p)(3q-1) & 3p(1-p) \\ -3q(1-q) & 1+(1-2q)(2-3p) \end{bmatrix}$$

Evaluated at ($p^* = 2/3$, $q^* = 1/3$), $J = \begin{bmatrix} 1 & 2/3 \\ -2/3 & 1 \end{bmatrix}$, which has eigenvalues equal to $1 \pm (2/3)i$

Because $\text{abs}(1 \pm (2/3)i) = 1.444 > 1$, the steady state is unstable.